

Quantum tricriticality and quantum phases in the Lipkin-Meshkov-Glick model

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We analyse the Lipkin-Meshkov-Glick model near, but not in, the thermodynamic ($N \rightarrow \infty$) limit, for arbitrary strengths of the two types of inter-fermion interaction, and at non-zero temperature. We use semiclassical methods to obtain the crossover temperature, $T^*(V, W)$, between linear and nonlinear behavior. It is shown that T^* typically vanishes logarithmically as the quantum phase transition from the symmetric to the distorted state is approached, except near the quantum tricritical point $(V, W) = (0, -\varepsilon)$, where it becomes linear. We discuss the relationship of the T^* line to those observed in conventional condensed matter systems. These calculations are supplemented by a direct quantum solution near the line $V = 0$. This reveals a quantum singlet phase for large positive W that is not accessible via semi-classical techniques.

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Introduction. The Lipkin-Meshkov-Glick (LMG) model has long been the subject of study in the nuclear physics community. It was introduced in 1959 by Fallieros [1] as a model of monopole oscillations in the ^{16}O nucleus, and subsequently studied by Volkov [2], before springing to prominence with the work of Lipkin, Meshkov, and Glick in 1965 [3–5], who considered it as a non-trivial correlated model against which various approximation schemes could be tested.

The Hamiltonian of the LMG model is:

$$H = \frac{\varepsilon}{2} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} + \frac{\tilde{W}}{2} \sum_{pq\sigma} a_{p\sigma}^\dagger a_{q\bar{\sigma}}^\dagger a_{q\sigma} a_{p\bar{\sigma}} + \frac{\tilde{V}}{2} \sum_{pq\sigma} a_{p\sigma}^\dagger a_{q\sigma}^\dagger a_{q\bar{\sigma}} a_{p\bar{\sigma}}, \quad (1)$$

where the $a_{p\sigma}$ are fermionic annihilation operators, $\sigma = \pm 1$ is a spin-like index denoting the nuclear shell in which the fermion is, $\bar{\sigma}$ represents the opposite spin to σ , and $p = 1, 2, \dots, N_p$ is an auxiliary quantum number distinguishing between a large number of degenerate levels within each shell. The \tilde{W} interaction represents an exchange of particles between the lower- and higher-energy shells, while \tilde{V} represents pair-tunnelling between shells (a sort of Josephson term). This Hamiltonian clearly conserves the number of particles, i.e.

$$N \equiv \sum_{p\sigma} a_{p\sigma}^\dagger a_{p\sigma} \quad (2)$$

is a good quantum number. It is sometimes stipulated that $N = N_p$ (the ‘half-filled’ case), but we shall consider all possible values of N .

The literature on the LMG model is extensive [6]. The zero-temperature phase diagram has been obtained in the thermodynamic limit [7–9] and finite-size corrections analysed [7, 10]. Non-zero-temperature properties of the model have also been studied [11], as have properties of

the zero-temperature entanglement entropy [12] and the negativity [13]. Frequently these studies confine themselves to a particular line or region in the two-parameter space (\tilde{V}, \tilde{W}) [14, 15], but some works consider the whole plane. It should also be noted that the $\tilde{V} = 0$ line of the model is equivalent to the Dicke model [16], about which much is known [17–19].

However, there has to our knowledge not yet been a study of the finite- N , non-zero-temperature properties of the model over the whole (\tilde{V}, \tilde{W}) -plane. In this Letter we carry out that study, with particular emphasis on the nature of the crossover from linear to nonlinear dynamics that occurs as the quantum phase transitions in the (\tilde{V}, \tilde{W}) -plane are approached at non-zero temperature.

The large- N_p limit must be taken with care. Since the interaction terms in (1) cause every level to interact equally with every other (producing a result $\sim N_p^2$ provided that N/N_p is finite), we must compensate by sending \tilde{V} and \tilde{W} to zero in the following way:

$$\tilde{V} = \frac{V}{N_p}, \quad \tilde{W} = \frac{W}{N_p}, \quad (3)$$

with V and W held constant as $N_p \rightarrow \infty$. Then the energy remains proportional to N_p , with sub-dominant corrections $O(1)$. However, as we shall see below, these sub-dominant pieces are crucial in breaking a degeneracy in N_p in certain regions of the ground-state phase diagram.

A key observation, made in the original paper [3], is that the Hamiltonian (1) may be rewritten in terms of the following pseudospin operators:

$$J_z \equiv \frac{1}{2} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma}, \quad J_+ \equiv \sum_p a_{p1}^\dagger a_{p\bar{1}}, \quad J_- = J_+^\dagger. \quad (4)$$

These obey the standard angular momentum commutation relations. (Note that we have adopted units in which $\hbar = 1$.) The pseudospin version of the Hamiltonian may

easily be shown to be:

$$H = \varepsilon J_z + \frac{V}{2N_p} (J_+^2 + J_-^2) + \frac{W}{2N_p} (J_+ J_- + J_- J_+ - N), \quad (5)$$

where N is the particle-number operator defined in (2). This form of the Hamiltonian is most useful for direct quantum treatments; for classical and semi-classical approaches it is preferable to rewrite it again using the definitions $J_{\pm} = J_x \pm iJ_y$:

$$H = \varepsilon J_z + \frac{V}{N_p} (J_x^2 - J_y^2) + \frac{W}{N_p} \left(J_x^2 + J_y^2 - \frac{N}{2} \right). \quad (6)$$

As well as the particle number, N , this Hamiltonian also clearly conserves the magnitude of the pseudospin:

$$\mathbf{J}^2 \equiv J_x^2 + J_y^2 + J_z^2 = J(J+1), \quad (7)$$

where J is an integer between 0 and $N/2$. (We assume here and henceforth that N and N_p are even.)

In the remainder of this Letter, we shall summarize the existing classical and semiclassical analysis of the model, and extract from the latter an explicit form for the crossover temperature $T^*(V, W)$. Then we shall show, by solving the model beyond the semiclassical approximation near the $V = 0$ line, that the ground-state phase diagram contains at least one purely quantum phase, not captured by the semiclassical analysis.

Classical phase diagram. Firstly we shall summarize the analysis of the Hamiltonian (6) in the limit $J, N \rightarrow \infty$ [7]. Note that these limits, while not independent, are nonetheless not the same: $J \rightarrow \infty$ implies $N \rightarrow \infty$, but not vice versa. In the $J \rightarrow \infty$ limit, the operators may be replaced by classical vectors, and the energy minimized straightforwardly. We choose the following parameterisation:

$$J = \frac{J_{\max}}{2} (1 - \cos \alpha), \quad (8)$$

$$(J_x, J_y, J_z) = (J \sin \theta \cos \phi, J \sin \theta \sin \phi, J \cos \theta). \quad (9)$$

The maximum possible value of J , denoted J_{\max} , is of course a function of N :

$$J_{\max} = \begin{cases} \frac{N}{2} & 0 \leq N \leq N_p, \\ N_p - \frac{N}{2} & N_p < N \leq 2N_p. \end{cases} \quad (10)$$

With this parameterisation, the energy as a function of α , θ , and ϕ is found to be

$$E = \frac{J_{\max}}{2} (1 - \cos \alpha) \left[\varepsilon \cos \theta + \frac{W}{4} (1 - \cos \alpha) \sin^2 \theta + \frac{V}{4} (1 - \cos \alpha) \sin^2 \theta \cos 2\phi \right]. \quad (11)$$

Minimising this energy, one obtains the ground state phase diagram shown in Fig. 1. The three phases shown

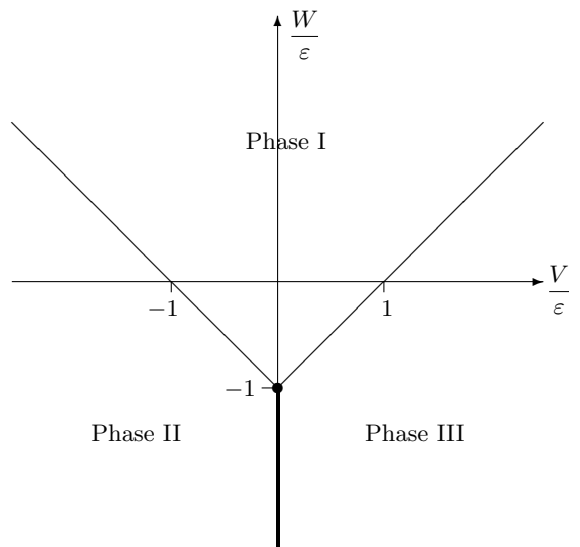


FIG. 1: The ground state phase diagram of the LMG model in the classical ($N, J \rightarrow \infty$) limit. The bold transition line at $V = 0$ is first-order, while the other two are second-order; they meet at a tricritical point when $(V, W) = (0, -\varepsilon)$.

in the diagram are characterized by the following behaviors of the pseudospin \mathbf{J} :

Phase I. Full spin, oriented in the negative z direction. This corresponds to the parameter values $\alpha = \pi$, $\theta = \pi$, ϕ undetermined. This ground state is non-degenerate.

Phase II. Full spin, canting from the negative z direction (near the transition line) to either the negative or positive x direction (deep in the phase). This corresponds to the parameter values $\alpha = \pi$, $\phi = 0$ or π , and

$$\theta = \arccos \left(\frac{\varepsilon}{W + V} \right). \quad (12)$$

This shows in particular that the transition is second-order: there is no jump in the spin's angle of orientation as the boundary between phases I and II is crossed. In phase II, the ground state is always doubly degenerate.

Phase III. As phase II, but with $\phi = \pm\pi/2$.

The phase boundary between phases II and III is first-order, since while α and θ are continuous across it, it involves a discontinuous jump of the parameter ϕ , corresponding to a reorientation of the spin from the x - to the y -axis. However, it is a peculiar sort of first-order transition, since at the transition all values of ϕ become degenerate, and hence despite being first-order it does have associated soft modes. This emergent U(1) symmetry at $V = 0$ is just the phase of the coherent photon field in the superradiant phase of the Dicke model, with the quantum tricritical point at $(V, W) = (0, -\varepsilon)$ corresponding to the superradiance transition.

Crossover to non-linear dynamics. We now proceed to analyse the finite- N model at non-zero temperature. It

is natural to choose a Holstein-Primakoff representation [21–23] of the pseudospin, which we define with respect to phase I, i.e. with reference to a full spin oriented in the negative z -direction:

$$J_z \equiv -J + b^\dagger b, \quad J_+ \approx \sqrt{2J} b^\dagger, \quad J_- \approx \sqrt{2J} b, \quad (13)$$

where the boson operators b and b^\dagger obey the usual commutation relations $[b, b^\dagger] = 1$, and the commonly used linear approximation has been made. Although J is formally variable, we shall here treat it as fixed at $N_p/2$.

Substituting this approximation into (5), and applying the commutation relations for the b -operators, we obtain that

$$H = -\frac{\varepsilon N_p}{2} + (\varepsilon + W) b^\dagger b + \frac{V}{2} ((b^\dagger)^2 + (b)^2). \quad (14)$$

It is clear from (14) that the tricritical point (at $V = 0$) corresponds to the point where the boson energy becomes negative, signalling an instability which mathematically invalidates the linear approximation, and physically corresponds to the superradiance transition. To extend the analysis to non-zero V , we must make a Bogolyubov rotation [24] to eliminate the anomalous terms b^2 and $(b^\dagger)^2$; this rotation is given by

$$b = \beta \cosh \eta + \beta^\dagger \sinh \eta, \quad (15)$$

$$b^\dagger = \beta \sinh \eta + \beta^\dagger \cosh \eta, \quad (16)$$

which has been designed to guarantee that $[\beta, \beta^\dagger] = 1$, i.e. that the new operators are also bosons. In terms of these, the Hamiltonian (14) becomes

$$\begin{aligned} H = & ((\varepsilon + W) \cosh(2\eta) + V \sinh(2\eta)) \beta^\dagger \beta \\ & + \frac{1}{2} ((\varepsilon + W) \sinh(2\eta) + V \cosh(2\eta)) (\beta^2 + (\beta^\dagger)^2) \\ & + \text{const.} \end{aligned} \quad (17)$$

The angle η that eliminates the anomalous terms is given by

$$\tanh(2\eta) = -\frac{V}{\varepsilon + W}. \quad (18)$$

Note that this equation has a solution only in the interval $|\varepsilon + W| \geq |V|$, i.e. in the area labelled ‘phase I’ in the classical analysis above. In the cases in which the Bogolyubov rotation is possible, the resulting Hamiltonian is

$$H = E \beta^\dagger \beta + \text{const.}, \quad (19)$$

where the boson energy E is given by

$$E = \sqrt{(\varepsilon + W)^2 - V^2}. \quad (20)$$

It is a familiar feature of quantum critical theories [25, 26] that the approach to a quantum critical point

at non-zero temperature is accompanied by a crossover from ‘renormalized classical’ to ‘quantum critical’ behavior. A phenomenon of the same sort takes place here: the dynamics of the model cross over from being approximately linear to fully nonlinear as the transition line is approached. A simple way to obtain the location of this crossover is to ask at what temperature the linear approximation (i.e. the condition that $\langle b^\dagger b \rangle \ll N_p$) breaks down. The temperature at which this happens is approximately given by $\langle b^\dagger b \rangle_T = N_p$; inserting the expressions for b and b^\dagger in terms of β and β^\dagger this becomes

$$\frac{\varepsilon + W}{\sqrt{(\varepsilon + W)^2 - V^2}} \left(\langle \beta^\dagger \beta \rangle_T + \frac{1}{2} \right) = N_p + \frac{1}{2}. \quad (21)$$

The value of the thermal average follows directly from the Bose-Einstein distribution, so that

$$\frac{\varepsilon + W}{\sqrt{(\varepsilon + W)^2 - V^2}} \left(\frac{1}{e^{\beta E} - 1} + \frac{1}{2} \right) = N_p + \frac{1}{2} \quad (22)$$

which yields the temperature

$$T^* = \frac{\varepsilon}{k_B} \frac{\sqrt{y^2 - x^2}}{2} \left[\text{atanh} \left(\frac{\sqrt{\gamma^2 - 1}}{\gamma} \frac{y}{\sqrt{y^2 - x^2}} \right) \right]^{-1}. \quad (23)$$

In this last line we have made several useful definitions:

$$x \equiv \frac{V}{\varepsilon}, \quad y \equiv 1 + \frac{W}{\varepsilon}, \quad \gamma \equiv \left(1 - \frac{1}{(2N_p + 1)^2} \right)^{-1/2}. \quad (24)$$

Notice that in the (x, y) co-ordinate system the tricritical point is at the origin.

The first thing to observe about (23) is that it vanishes not at the original phase transition $y = x$ but at $y = \gamma x$. This is a renormalization of the position of the transition line due to quantum fluctuations, similar to those discussed in [27]. To examine the behavior of T^* as this renormalized transition line is approached, we set $y = \gamma x + \delta$, with $0 < \delta \ll 1$. In this limit, provided that $x \gg \delta$, we obtain

$$T^* \approx \frac{\varepsilon}{k_B} \frac{x \sqrt{\gamma^2 - 1}}{\ln(2\gamma(\gamma^2 - 1)x) - \ln \delta} \sim -\frac{1}{\ln \delta} \quad (25)$$

as $\delta \rightarrow 0^+$. Hence this second-order transition has an extremely narrow quantum critical cone, in contrast with the simple power laws typically observed in quantum critical theories [25, 26]. The expression (25) becomes invalid as the tricritical point at $(x, y) = (0, 0)$ is approached. It crosses over to a much simpler behavior, which may be obtained by setting $x = 0$ and then taking $0 < y \ll 1$:

$$T^* \approx \frac{\varepsilon}{k_B} \frac{y}{2} \left[\text{atanh} \left(\frac{\sqrt{\gamma^2 - 1}}{\gamma} \right) \right]^{-1} \sim y \quad (26)$$

as $y \rightarrow 0^+$. Thus, perhaps surprisingly, power-law behavior is recovered at the tricritical point $y = x = 0$

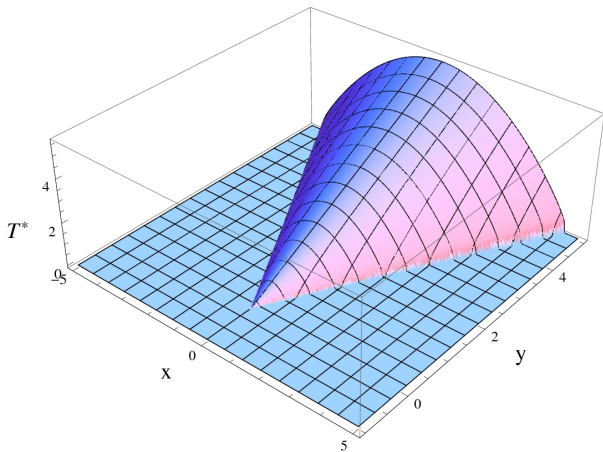


FIG. 2: The crossover temperature T^* as a function of x and y , the rescaled and offset interaction parameters of the LMG model. The quantum tricritical point is at the origin; note the crossover from logarithmic to linear behavior of T^* as this point is approached. The line $x = 0$ corresponds to the Dicke model.

despite being absent along the rest of the critical line $y = \gamma x$. Analysis of the transition from the other phases may be carried out in a similar fashion, using the form of the Holstein-Primakoff transformation appropriate to the phase in question. The full behavior of (23) is shown in Fig. 2.

Additional quantum phases. To go beyond the semiclassical analysis, and thus describe the behavior at temperatures $T > T^*$, requires in principle a full quantum solution of the model. While this is beyond the scope of this Letter, it is appropriate to give a brief demonstration that the quantum solution of the LMG model contains at least one phase that does not appear in the classical phase diagram [20]. To expose it, let us slightly rewrite the Hamiltonian (6) with $V = 0$ as

$$H = \varepsilon J_z + \frac{W}{N_p} \left(\mathbf{J}^2 - J_z^2 - \frac{N}{2} \right). \quad (27)$$

Classically, it is possible to make $\mathbf{J}^2 - J_z^2$ zero by orienting the spin along the positive or negative z -direction; quantum mechanically, however, this zero is achieved only in the singlet state, where $\mathbf{J}^2 = 0$.

This effect manifests itself at large positive values of W/ε , where to a first approximation one may neglect the ε term entirely. The Hamiltonian is then minimized by (a) choosing a singlet state for the pseudospin, and (b) manufacturing this singlet state from the largest possible number of particles; in this case, that number is $N = 2N_p$, corresponding to full occupation of all the levels in the original LMG model. The energy of this singlet state is therefore simply $E_{\text{singlet}} = -W$. By comparison, the energy of the phase I state at $V = 0$ is $E_{\text{full-spin}} = -\varepsilon N_p/2$. Hence the transition from phase I

to the singlet phase occurs when $W/\varepsilon = N_p/2$. It thus recedes to infinite W in the thermodynamic limit, but is present at large W for any finite N_p . (Further analysis shows that there are no intervening phases; the transition occurs directly from full- to zero-spin, though via a rather interesting quantum critical point.)

For non-zero V , quantum fluctuations shift the critical value of W ; to leading non-zero order in perturbation theory, the resulting behavior is given by:

$$W_c = \frac{\varepsilon N_p}{2} + \frac{V^2}{\varepsilon N_p} \left(1 - \frac{1}{N_p} \right). \quad (28)$$

The fact that W_c increases with increasing V represents a pseudo-entropic favouring of the full-spin state, of the type recently discussed by Conduit *et al.* [28].

Summary. In this Letter, we have considered the non-zero-temperature behavior of the Lipkin-Meshkov-Glick model at large but finite N , with specific emphasis on the crossover from linear to nonlinear dynamics in the vicinity of the model's zero-temperature quantum phase transitions. We have obtained the functional form of this crossover surface, above which a full quantum treatment of the model is necessary. We have also demonstrated the importance of a full quantum treatment at large W , where a singlet phase emerges for finite values of N_p . Further work should be done to explore the properties of the region $T > T^*$, particularly near the quantum tricritical point. This is especially relevant given the importance of tricritical points in the theory of nearly ferromagnetic metals [29].

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